Math 259A Lecture 16 Notes

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November 4, 2019

1 Examples of Factors

1.1 Type I factors

Last time, we discussed the classification of von Neumann algebras by type. The proof is an exercise; it consists of taking maximal orthogonal projections of each type, one type at a time, and looking at the rest of the space.[1]

Proposition 1.1. *If $\{x_i\}_i \subseteq (M)_1$ with mutually orthogonal central supports, then $\sum_i x_i$ is a SO-convergent sum. In fact, if $\{\ell(x_i)\}_i$ are mutually orthogonal, $\{r(x_i)\}_i$ are mutually orthogonal, then $\sum_i x_i$ is SO-convergent.*

We have these 5 types of von Neumann algebras, but we are really interested in factors.

Definition 1.1. A **factor** is a von Neumann algebra M with $Z(M) = \mathbb{C}$.

Example 1.1. Type I finite factors are algebras with $M \cong M_{n\times n}(\mathbb{C}) = \mathcal{B}(\ell^2_n)$.

Example 1.2. Type I infinite factors have $M \cong \mathcal{B}(\ell^2(I))$ for some infinite I.

Lemma 1.1. *If M is a type I factor and $e \in M$ is abelian, then $eMe \cong \mathbb{C}e$.*

Proof. Consider e and $1-e$. We must have $e \succ 1-e$ or $e \prec 1-e$, We can't have the former, so $e \prec 1-e$. Now repeat this with $e_2 \leq 1-e$. We can then find a maximal projection like this. $\square$

1.2 Group von Neumann algebras

Let Γ be a discrete group (not necessarily countable), and let $\lambda : \Gamma \to \mathcal{B}(\ell^2(\Gamma))$ be the left regular representation: $\lambda(g)(\xi_h) = \xi_{gh}$. We can also take the right regular representation $\rho : \Gamma \to \mathcal{B}(\ell^2(\Gamma))$ given by $\rho(g)\xi_h = \xi_{hg^{-1}}$. We have that span $\lambda(\Gamma)$ is a *-algebra, so its weak closure is a von Neumann algebra.

[1] Professor Popa said this would be an exercise and then proceeded to write out the proof, which follows this skeleton. I got too lazy to copy down the definition of each individual projection.

Definition 1.2. We call $L(\Gamma) := \overline{\operatorname{span} \lambda(\Gamma)}^{\mathrm{wk}} = \lambda(\Gamma)''$ the **group von Neumann algebra** of Γ. Similarly, we have $R(\Gamma) = \rho(\Gamma)''$. We have $[\lambda(\Gamma), \rho(\Gamma)] = 0$, so $L(\Gamma), R(\Gamma)] = 0$.

Define $\tau : L(\Gamma) \to \mathbb{C}$ by $\tau(x) = \langle x\xi_e, \xi_e \rangle$. Notice that

$$\tau(\lambda(g)\lambda(h)) = \langle \xi_{gh}, \xi_e \rangle = \delta_{gh,e} = \delta_{hg,e} = \tau(\lambda(h)\lambda(g)).$$

So $\tau(xy) = \tau(yx)$ for all $x, y \in L(\Gamma)$. Also, τ is a state, and it satisfies the **traciality property** $\tau(1) = 1$. τ is faithful ($\tau(x^*x) = 0 \implies x = 0$), since ξ_e is separating for $L(\Gamma)$.

Proposition 1.2. *If a von Neumann algebra M has a faithful trace, then M is finite.*

Proof. If $u^*u = 1$, then $\tau(1 - uu^*) = 1 - \tau(uu^*) = 1 - \tau(u^*u) = 0$. So $uu^* = 1$. $\square$

Corollary 1.1. *$L(\Gamma)$ is finite.*

When is $L(\Gamma)$ a factor?

Theorem 1.1. *$L(\Gamma)$ is a factor if and only if Γ is infinite conjugacy class (i.e. for any $g \neq e$, $\{hgh^{-1} : h \in \Gamma\}$ is infinite).*

Proof. ($\implies$): Assume there exists some $g \neq e$ such that $\{hgh^{-1} : h \in \Gamma\}$ is finite. This is $\{g_1, \dots, g_n\} \not\ni e$. Let $z = \sum_{i=1}^n \lambda(g_i)$. Then $\lambda(h) z \lambda(h^{-1}) = z$, and $z(\xi_e) = \sum_{i=1}^n \xi_{g_i} \perp \xi_e$. But since ξ_e is separating, $z \in Z(M)$ and is not a scalar.

($\impliedby$): If we have z with $z(\xi_e) = \sum c_g \xi_g \in \ell^2$ with $z \notin \mathbb{C}1$, then there exists some $g_0 \neq e$ with $c_{g_0} \neq 0$. If $u_h z u_h^* = z$, then $c_{g_0} = c_{hg_0h^{-1}}$ for all h. $\square$